Intro to Algebra Syllabus and Prerequisites

TuTuring

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This course is meant to prepare students for Algebra 1 or similar middle school math classes. Prerequisite knowledge for this class can be found from page 2 on.

This course consists of ten 1.5-hour lessons and corresponding homework assignments. Extra practice beyond homework may be available. The topic schedule is below (subject to change).

Day	Topic
1	Variables
2	Expressions and Equations
3	Linear Equations
4	Quadratics
5	Functions
6	Graphing
7	Other Operations
8	Inequalities
9	Sequences
10	Number Theory

Contents

1 Preface

Before beginning the Intro to Algebra course, please make sure that you are at minimum somewhat familiar with the following topics. This document may double as a cheat sheet in case necessary. If you have any questions, please email us at tuturingtutoring@gmail.com.

2 Arithmetic

2.1 Arithmetic operations

1. Addition

   1.1. In $x + y = z$, x and y are the **addends**, and z is the **sum**.

   1.2. $x + 0 = x$

2. Subtraction

   2.1. In $x - y = z$, x is the **minuend**, y is the **subtrahend**, and z is the **difference**. The terms minuend and subtrahend are rarely used.

   2.2. $x - 0 = x$

3. Multiplication

   3.1. In algebra, the dot $\cdot$ is used for multiplication in place of the times symbol $\times$.

   3.2. In $x \times y = x \cdot y = z$, x and y are the **factors**, and z is the **product**.

   3.3. $x \cdot 1 = x$

   3.4. $0 \cdot x = 0$

4. Division

   4.1. In $x \div y = z$, x is the **dividend**, y is the **divisor**, and z is the **quotient**.

   4.2. $x \div 1 = x$

   4.3. $0 \div x = 0$

5. Exponentiation

   5.1. $x^y = x \cdot x \cdot x \ldots$ (**exponentiation** is **repeated multiplication**, so x is repeated y times)

   5.2. In $x^y = z$, x is the **base** and y is the **exponent**. We read this as "x to the **power** of y".

   5.3. $x^1 = x$

   5.4. If $x \neq 0$, $x^{-y} = 1 \div (x^y)$

   5.5. If $x \neq 0$, $x^0 = 1$

   5.6. $x^y \cdot x^z = x^{y+z}$

   5.7. $(x^y)^z = x^{y \cdot z}$

   5.8. $x^{y^z} = x^{(y^z)}$

2.2 Negative numbers

1. The **negation** (or **additive inverse**) of x is found by subtracting from zero: $0 - x = -x$. This is why subtraction is the **inverse operation** for addition.

2. Adding $-x$ is equivalent to subtracting x: $y + (-x) = y - x$

3. The product of a positive number and a negative number is negative: $x \cdot (-y) = -(x \cdot y)$

4. The product of two negative numbers is positive: $(-x) \cdot (-y) = x \cdot y$

2.3 Reciprocals

1. The **reciprocal** (or **multiplicative inverse**) of x is found by dividing from one: $1 \div x = \frac{1}{x}$. This is why division is the **inverse operation** for multiplication.
2. Multiplying by $\frac{1}{x}$ is equivalent to dividing by x: $y \cdot \frac{1}{x} = y \div x$
3. Multiplying a number by its reciprocal equals 1: $x \cdot \frac{1}{x} = 1$

2.4 Properties of arithmetic operations

Different operations have different properties from the list below.
For instance, addition is **commutative**: $1 + 2 = 2 + 1$, while division is not: $1 \div 2 \neq 2 \div 1$.

1. **Commutative property:** $x \star y = y \star x$
   1.1. Addition and multiplication have this property.
2. **Associative property:** $(x \star y) \star z = x \star (y \star z)$
   2.1. Addition and multiplication have this property.
3. **Identity property:** $x \star I = x$
   3.1. Addition, subtraction, multiplication, division, and exponentiation all have this property.
   3.2. I is known as the **identity element**.
   3.3. For addition and subtraction, the **identity element** is 0. Adding or subtracting 0 from any number does not change its value.
   3.4. For multiplication and division, the **identity element** is 1. Multiplying or dividing any number by 1 does not change its value.
   3.5. For exponentiation, the **identity element** is 1. Raising a number to the **power** of 1 does not change its value.
4. **Inverse property:** $x \star x' = I$
   4.1. Addition and multiplication have this property, while subtraction and division are the inverse operations of addition and multiplication.
   4.2. x' equals the **inverse** of x under the operation $\star$.
   4.3. The **additive inverse** of x is $-x$
   4.4. The **multiplicative inverse** of x is $\frac{1}{x}$
5. **Distributive property:** $x \star (y * z) = (x \star y) * (x \star z)$
   5.1. The distributive property connects two operations. For multiplication and addition, multiplication takes the place of $\star$ and addition takes the place of $*$.
   5.2. Since division is the **inverse** of multiplication and subtraction is the **inverse** of addition, both multiplication and division are **distributive on** addition and subtraction.

3 Exponents and roots

Remember, exponentiation is repeated multiplication: $x^y = x \cdot x \cdot x \ldots$ (x is repeated y times). The **root** is the **inverse operation** of exponentiation.

3.1 Squares and cubes

1. The **square** of a number x is the second **power** of x: equal to x^2, or $x \cdot x$
   1.1. The **square** of x also equal to the **area** of a square with side length x
   1.2. If a number y is equal to the **square** of some other **whole number** x ($y = x^2$), then y is called a **perfect square**.
2. The **cube** of a number x is the third **power** of x: equal to x^3, or $x \cdot x \cdot x$
   2.1. The **cube** of x also equal to the **volume** of a cube with side length x
   2.2. If a number y is equal to the **cube** of some other **whole number** x ($y = x^2$), then y is called a **perfect cube**.

3.2 Roots

The **root** is the **inverse operation** of exponentiation: if $z^y = x$, then $\sqrt[y]{x} = z$

1. In $\sqrt[y]{x}$, x is the **radicand**, y is the **index**, and $\sqrt[y]{x}$ is the **radical**.
2. The **root** can also be written as an **exponent**: $\sqrt[y]{x} = x^{\frac{1}{y}}$

3.3 Square and cube roots

Square roots show up often in algebra, especially in the context of **quadratic equations**.

1. The **square root** of a number x is equal to $\sqrt[2]{x}$: $\sqrt[2]{x} \cdot \sqrt[2]{x} = x$
2. The **cube root** of a number x is equal to $\sqrt[3]{x}$: $\sqrt[3]{x} \cdot \sqrt[3]{x} \cdot \sqrt[3]{x} = x$

4 Divisibility and prime numbers

4.1 Multiples

1. If z equals x times some integer ($x \cdot y = z$), z is a **multiple** of x.
2. If y is a multiple of x, then any multiple of y is a multiple of x.

4.2 Divisibility

1. In the inverse, if $z \div x$ equals a whole number ($z \div x = y$), z is **divisible** by x.
2. If y is divisible by x, then any number divisible by y is divisible by x.

4.3 Divisibility rules

Some numbers have simple divisibility rules in **base 10** (see **base 10** in the **Decimals** section).

1. All whole numbers are divisible by 1.
2. Since 10 is divisible by two, only the ones place matters. All even numbers are divisible by 2.
3. All numbers where the sum of digits is divisible by three are divisible by 3.
4. Since 100 is divisible by four, only the tens and ones place matters. All numbers where the last two digits are multiples of four are divisible by 4.
5. Since 10 is divisible by five, only the ones places matters. All numbers ending in zero or five are divisible by 5.
6. All numbers divisible by two and three are divisible by 6.
8. Since 1000 is divisible by eight, only the last 3 places matters. All numbers where the last two digits are multiples of eight are divisible by 8.
9. All numbers where the sum of digits is divisible by nine are divisible by 9.

4.4 Prime and composite numbers

Prime numbers are numbers that are only **divisible** by two integer factors: 1 and itself. **Composite numbers** have more than two integer factors.

1. 1 is not a prime number.
2. Small prime numbers include: 2, 3, 5, 7, 11, 13, 17, 23...

4.5 Prime factorization

The **prime factorization** of a integer is a **product** of **prime numbers** that equal the integer.

1. When an integer is composite, it can be expressed as a product of two or more integers that are greater than one.
2. The prime factorization repeats the above process until all terms in the product are prime.
3. For instance, the prime factorization of 60 is $60 = 2 \cdot 2 \cdot 3 \cdot 5 = 2^2 \cdot 3 \cdot 5$

4.6 Greatest common divisor

The **greatest common divisor** (GCD) of two numbers is the largest integer that is a divisor of both numbers.

1. The GCD of 72 and 60 is 12: $72 \div 12 = 6$ and $60 \div 12 = 5$. Since 6 and 5 do not share any factors, we can be sure that this is true.

2. The GCD can be found by comparing prime factorizations and finding common prime factors. $72 = 2^3 \cdot 3^2$ and $60 = 2^2 \cdot 3 \cdot 5$, so the GCD can have at most 2 **powers** of 2 and 1 **power** of 3. $2^2 \cdot 3 = 12$, which is indeed the GCD of 72 and 60.

3. If the GCD of two numbers is 1, they are **coprime**.

4.7 Least common multiple

The **least common multiple** (LCM) of two numbers is the smallest integer that is **divisible** by both numbers.

1. The LCM of 12 and 10 is 60: $60 \div 12 = 5$ and $60 \div 10 = 6$. Since 5 and 6 do not share any factors, we can be sure that this is true.

2. The LCM can be found by comparing prime factorizations and finding missing prime factors. $12 = 2^2 \cdot 3$ and $10 = 2 \cdot 5$, so 12 is missing a factor of 5 and 10 is missing a factor of 2 and a factor of 3. $12 \cdot 5 = 10 \cdot 2 \cdot 3 = 60$, which is indeed the LCM of 12 and 10.

3. Since the LCM doesn't want to overlap any more factors than it needs to, we can also find the LCM of x and y like so: $LCM(x, y) = x \cdot y \div GCD(x, y)$

4. If the LCM of two numbers equals their product, they are **coprime**.

5 Fractions

Fractions are often used in place of the division symbol. The quotient $x \div y$, where y is nonzero, can be written $x \div y = \frac{x}{y}$.

Fractions are used to denote parts of a whole. For instance, $\frac{2}{3}$ means 2 out of 3 equal parts. If you divided a cake into 3 equal pieces, two of those pieces would equal $\frac{2}{3}$ of the cake.

5.1 Fraction basics

1. In $\frac{x}{y}$, x is the **numerator**, y is the **denominator**, and the two are separated by a line called the **fraction bar**.
2. If $x \neq 0$, $\frac{0}{x} = 0$ and $\frac{x}{x} = 1$
3. If $y \neq 0$, $\frac{-x}{y} = \frac{x}{-y} = -\frac{x}{y}$ and $\frac{-x}{-y} = \frac{x}{y}$
4. If $x \neq 0$ and $y \neq 0$, $\frac{1}{\frac{x}{y}} = \frac{y}{x}$ (**reciprocal**)
5. If $x > y$, $\frac{x}{y}$ is called an **improper fraction**.
6. **Mixed numbers** have both a **whole number** and a **proper fraction**, representing the sum of the two: $x\frac{y}{z} = x + \frac{y}{z}$. However, this notation is not preferred.

5.2 Fraction operations

1. **Addition and subtraction:**
   1.1. Fractions can be added or subtracted when their **denominators** are equal: $\frac{x}{z} + \frac{y}{z} = \frac{x+y}{z}$
   1.2. To add two fractions with different denominators, we can multiply them by $1 = \frac{y}{y} = \frac{z}{z}$:
   $\frac{x}{y} + \frac{w}{z} = (\frac{x}{y} \cdot 1) + (1 \cdot \frac{w}{z}) = (\frac{x}{y} \cdot \frac{z}{z}) + (\frac{y}{y} \cdot \frac{w}{z}) = \frac{x \cdot z}{y \cdot z} + (\frac{y \cdot w}{y \cdot z} = \frac{x \cdot z + y \cdot w}{y \cdot z})$
2. **Multiplication**
   2.1. Multiply the numerators and denominators respectively: $\frac{x}{y} \cdot \frac{w}{z} = \frac{x \cdot w}{y \cdot z}$
3. **Division**
   3.1. Divide the numerators and denominators respectively: $\frac{x}{y} \div \frac{w}{z} = \frac{x \div w}{y \div z}$
   3.2. Dividing by x is also equivalent to multiplying by its reciprocal $\frac{1}{x}$:
   $\frac{x}{y} \div \frac{w}{z} = \frac{x}{y} \cdot \frac{1}{\frac{w}{z}} = \frac{x}{y} \cdot \frac{z}{w} = \frac{x \cdot z}{y \cdot w}$
4. **Exponentiation**
   4.1. Exponentiation is repeated multiplication: $\frac{x}{y}^z = \frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y} \ldots$ (z times) $= \frac{x \cdot x \cdot x \ldots}{y \cdot y \cdot y \ldots} = \frac{x^z}{y^z}$

5.3 Simplifying fractions

Fractions are considered **simplified** if the numerator and denominator do not share any **factors** (also called **coprime**).

1. If $x \div z = a$ and $y \div z = b$, then a simpler version of $\frac{x}{y}$ is $\frac{x}{y} = \frac{x}{y} \div 1 = \frac{x}{y} \div \frac{z}{z} = \frac{x \div z}{y \div z} = \frac{a}{b}$

6 Decimals

6.1 Counting in base 10

1. From the ones digit, digits to the left increase in value by **powers** of ten.
2. 2538 can be written as $2538 = 2000 + 500 + 30 + 8 = 2 \cdot 10^3 + 5 \cdot 10^2 + 3 \cdot 10^1 + 8 \cdot 10^0$

6.2 Decimals

1. Digits to the right of the **decimal point** denote a non-whole part of a number.
2. From the decimal point, digits to the right decrease in value by **powers** of ten.
3. 2.538 can be written as $2.538 = 2 + 0.5 + 0.03 + 0.008 = 2 \cdot 10^0 + 5 \cdot 10^-1 + 3 \cdot 10^-2 + 8 \cdot 10^-3$

6.3 Decimal Arithmetic

1. **Addition and subtraction**
   1.1. Addition and subtraction work the same way as whole numbers. Make sure that the correct digits are aligned with each other.
   1.2. For example: $1.23+14.91 = (1+0.2+0.03)+(10+4+0.9+0.01) = 10+5+1.1+0.04 = 16.14$
2. **Multiplication and division**
   2.1. Multiplication and division also work the same way. However, we must be very mindful of where the decimal point should be.
   2.2. When multiplying whole numbers, we have to keep track of how many zeroes should be at the end of each number: $30 \cdot 200 = (3 \cdot 10^1) \cdot (2 \cdot 10^2) = (3 \cdot 2) \cdot (10^1 \cdot 10^2) = 6 \cdot 10^3 = 6000$
   2.3. Same goes for multiplying decimals: $0.4 \cdot 0.003 = (4 \cdot 10^{-1}) \cdot (3 \cdot 10^{-3}) = (4 \cdot 3) \cdot (10^{-1} \cdot 10^{-3}) = 12 \cdot 10^{-4} = 0.0012$

6.4 Rounding

1. If the digit below the digit you are rounding to is less than 5, round down: $3.49 \to 3$
2. If the digit below is greater than or equal to 5, round down: $3.5 \to 4$
3. For other digits, do the same: $3.16 \to 3.2$
4. However, do not round multiple times: $3.47 \to 3.5 \not\to 4$

7 Inequalities

Inequalities compare the value of numbers or expressions. **Greater than**, **less than**, and **equal to** are terms used in inequalities.

7.1 Types of inequalities

1. If $x < y$, then x is **less than** y: the inequality $2 < 5$ is true.
2. If $x \leq y$, then x is **less than or equal to** y: the inequalities $2 \leq 2$ and $5 \leq 2$ are true.
3. If $x > y$, then x is **greater than** y: the inequality $5 > 2$ is true.
4. If $x \geq y$, then x is **greater than or equal to** y: the inequalities $2 \geq 2$ and $5 \geq 2$ are true.

7.2 Absolute value

Absolute value determines the distance between a number and zero. This distance is always positive.

1. The absolute value of x is written $|x|$
2. If $x \geq 0$, then $|x| = x$
3. If $x < 0$, then $|x| = -x$
4. If an inequality has an absolute value in it, we must consider both cases: when the expression inside the absolute value is greater than or equal to zero, and when it is less than zero.